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Application of an incidence theorem for conics: Cauchy problem and integrability of the dCKP equation

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Abstract

We demonstrate that Pascal's classical theorem for hexagons inscribed in conics allows one to define in a compact manner maps which are governed algebraically by the integrable discrete CKP equation. A theorem for conics on oriented triangulated surfaces is used to construct a well-posed Cauchy problem for these dCKP maps. Moreover, the same theorem is exploited to construct in a purely geometric manner a Bäcklund transformation for dCKP maps. Thus, the integrability of dCKP maps and their underlying nonlinear soliton equation is shown to be encoded in an incidence theorem of projective geometry.

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1. Introduction

Integrable systems which exhibit the remarkable interaction properties of solitons are ubiquitous in mathematics and arise in such diverse areas as hydrodynamics, plasma and solid state physics as well as in general relativity [6, 12, 24]. The importance of solitons in current technological applications such as optical fibre communication systems and Josephson junction superconducting devices has been well documented [10, 13]. The (A)KP, BKP and CKP hierarchies and their multi-component analogues may be regarded as the fundamental hierarchies of integrable systems. Each scalar hierarchy may be recovered via sophisticated continuum limits from a *single* integrable discrete 'master' equation, namely, the dKP (Hirota) equation [11], the dBKP (Miwa) equation [16] and the dCKP equation, respectively [7].

It has recently come to light that there exist profound connections between fundamental incidence theorems of projective and conformal geometry and discrete integrable systems. For instance, Miquel's circle theorem has been shown to be central to the construction of both a well-posed Cauchy problem and a Bäcklund transformation for the discrete analogue

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of classical orthogonal coordinate systems in a three-dimensional Euclidean space (see [2] and references therein). The latter are governed by the Lamé system set down in 1840 which appears to be the oldest soliton system to be found in classical differential geometry [20]. Accordingly, it may be said that Miquel's theorem encapsulates the integrability of the discrete Lamé system underlying discrete orthogonal coordinate systems.

In [14], it has been demonstrated that Clifford's classical $\mathcal{C}_4$ point-circle configuration appropriately interpreted and extended to a lattice of fcc combinatorics encodes via Menelaus' theorem nothing but the Schwarzian version of the dKP equation. Similarly, Maxwell's theory of reciprocal figures developed in the context of graphical statics has been utilized to retrieve the Schwarzian BKP equation [15]. Furthermore, in [22], a generalization of Menelaus' theorem and Carnot's classical theorem have been shown to encode a Möbius invariant avatar of the dCKP equation.

In the present paper, we investigate in detail the projective geometry associated with the dCKP equation. Thus, we first recall the definition of discrete CKP maps as set down in [22] and demonstrate the fact that the constraints defining dCKP maps reduce to a *single* condition is a consequence of Pascal's classical theorem for hexagons inscribed in conics [18]. This complements the argument given in [22] which converted this geometric condition into an algebraic multi-ratio condition via Carnot's theorem. We then formulate a Cauchy problem for dCKP maps and prove its well posedness by employing an incidence theorem for conics on closed and oriented triangulated surfaces. This incidence theorem has been used in a different context to prove a 'non-Steinitz' theorem [19] and can be seen as a corollary of Carnot's theorem. Application of the incidence theorem also leads to the construction of a Bäcklund transformation which may be used to generate large classes of dCKP maps of arbitrary complexity. This illustrates in a purely geometric manner the integrable nature of the master dCKP equation.

2. The geometry of dCKP maps

In this section, we briefly review the geometric and algebraic properties of so-called *discrete CKP (dCKP) maps* as recorded in [22]. However, for convenience, we choose to adopt an equivalent geometric definition of dCKP maps which differs slightly from that presented in [22]. Thus, we are concerned with the geometry of 'lattices' in $\mathbb{R}^3$ indexed by the set of edges $\mathbb{E}$ of a cubic lattice $\mathbb{Z}^3$, that is, maps

$$\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3. \tag{2.1}$$

For notational convenience, we identify the edges of the cubic lattice with their centres $\boldsymbol{e}^i$, $i = 1, 2, 3$, and use the natural labelling

$$\boldsymbol{e}^1(\boldsymbol{n}) = \left(\tfrac{1}{2}, 0, 0\right) + \boldsymbol{n}, \qquad \boldsymbol{e}^2(\boldsymbol{n}) = \left(0, \tfrac{1}{2}, 0\right) + \boldsymbol{n}, \qquad \boldsymbol{e}^3(\boldsymbol{n}) = \left(0, 0, \tfrac{1}{2}\right) + \boldsymbol{n}, \tag{2.2}$$

where $\boldsymbol{n} \in \mathbb{Z}^3$. Any map $\boldsymbol{v}$ may therefore be regarded as a set of three maps $\boldsymbol{v}^i$ defined by

$$\boldsymbol{v}^i : \mathbb{Z}^3 \to \mathbb{R}^3, \qquad \boldsymbol{v}^i(\boldsymbol{n}) = \boldsymbol{v}(\boldsymbol{e}^i(\boldsymbol{n})). \tag{2.3}$$

2.1. Geometric description of dCKP maps

Particular maps $\boldsymbol{v}$ are obtained by imposing the following simple geometric condition.

Definition 1. *A map* $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$ *is termed a* discrete Darboux map *if it obeys the* collinearity condition, *that is, if the four images of the edges of any face of the cubic lattice are collinear.*

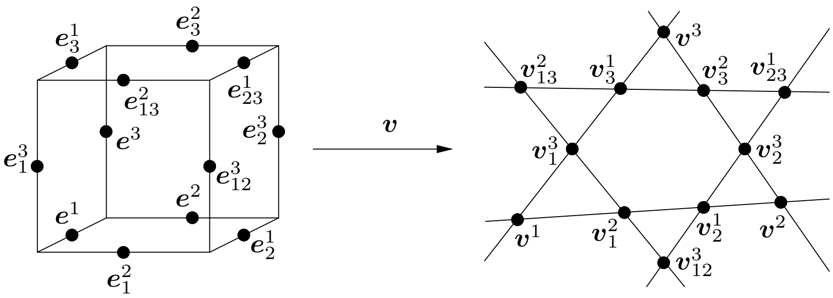

Figure 1. The geometry of a discrete Darboux map.

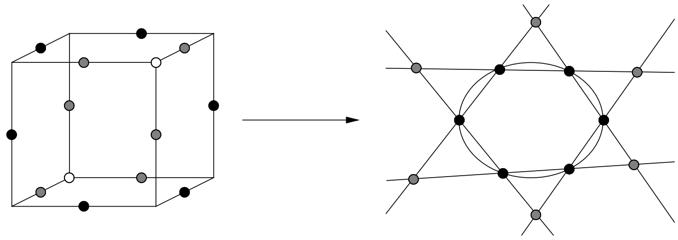

Figure 2. The geometry of a discrete CKP map.

By construction, a discrete Darboux map may be thought of as mapping the faces of the cubic lattice to lines in $\mathbb{R}^3$. It is evident that the six lines associated with the six faces of any elementary cube are coplanar and hence we may regard any elementary cube as being mapped to a plane in $\mathbb{R}^3$. The 12 edges of an elementary cube correspond to the 12 points of intersection of the associated six lines, excluding the points of intersection of the three pairs of 'opposite' lines. The latter correspond to the three pairs of 'opposite' faces of the elementary cube. The geometry of a discrete Darboux map is illustrated in figure 1. Here and in the following, subscripts denote unit increments of the discrete variables n_k so that, for instance,

$$g = g(n_1, n_2, n_3), \qquad g_1 = g(n_1 + 1, n_2, n_3), \qquad g_{23} = g(n_1, n_2 + 1, n_3 + 1). \tag{2.4}$$

In order to proceed, we now consider a vertex of an elementary cube and its 'opposite' counterpart as indicated by white vertices in figure 2. There exist six edges of the cube which are not connected to these two vertices. If the six images of these edges (black vertices in figure 2) lie on a conic or, equivalently, if the hexagon formed by the line segments connecting the six images is inscribed in a conic then we say that the corresponding *conic condition* is satisfied. Since the vertices of an elementary cube consist of four pairs of opposite vertices, there exist four conic conditions on any elementary cube. In [22], it has been established in an algebraic manner that if one conic condition on an elementary cube is satisfied then the three remaining conic conditions automatically hold. In fact, this assertion may readily be verified by employing the classical theorem of Pascal [18] which states that a hexagon is inscribed in a conic if and only if the three points of intersection of opposite edges (or their extensions) are collinear (cf figure 3). In the current context, this implies that if one conic condition on an elementary cube is satisfied then the points of intersection of the three pairs of opposite lines, which constitute the extended edges of the hexagon inscribed in the conic (cf figure 2), are collinear. The latter guarantees, in turn, that the other three hexagons generated by the six lines must also be inscribed in conics. Accordingly, it is admissible to refer to the four conic

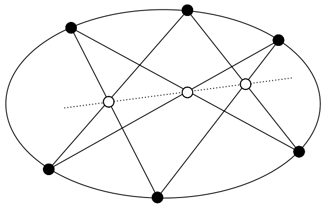

Figure 3. Pascal's theorem.

conditions as one conic condition defined on the cube. This forms the basis of the following definition.

Definition 2. *A discrete Darboux map* $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$ *is termed a* discrete CKP map *if the conic condition is satisfied on all elementary cubes of the cubic lattice.*

2.2. Algebraic description of dCKP maps

By definition, if a map $\boldsymbol{v}$ constitutes a discrete Darboux map then there exist six scalar functions ρ^{ik}, $i \neq k \in \{1, 2, 3\}$, corresponding to the collinearity conditions

$$\Delta_k \boldsymbol{v}^i = \rho^{ik}\big(\boldsymbol{v}_i^k - \boldsymbol{v}_k^i\big), \tag{2.5}$$

where the difference operators Δ_k are defined by $\Delta_k g = g_k - g$. Figure 1 illustrates the geometric content of the subset of equations (2.5) which prevails on a single elementary cube.

Conversely, the compatibility conditions $\Delta_l \Delta_k \boldsymbol{v}^i = \Delta_k \Delta_l \boldsymbol{v}^i$ associated with the linear lattice equations (2.5) produce the relations

$$\rho_l^{ik} = \big(1 + \rho_l^{ki}\big)\frac{(1+\rho^{kl})(1+\rho^{li})\rho^{ik} + (1+\rho^{ki})\rho^{lk}\rho^{il}}{(1+\rho^{ki})(1+\rho^{il}+\rho^{li})}, \tag{2.6}$$

where $i \neq k \neq l \neq i$. These may be solved for the quantities ρ_l^{ik} and therefore constitute a well-determined system for the functions ρ^{ik}. Accordingly, any set of functions ρ^{ik} obeying the above nonlinear system gives rise to a multiplicity of discrete Darboux maps $\boldsymbol{v}$. As shown in [22], this system is but another avatar of the well-known integrable discrete Darboux system which governs conjugate lattices in Euclidean space [4, 8]. However, it is emphasized that the connection between conjugate lattices and discrete Darboux maps as defined in the present paper occurs at the *nonlinear* level. In fact, at the linear (geometric) level and in the sense of soliton theory, discrete Darboux maps may be regarded as 'adjoint' to the maps defining conjugate lattices.

An alternative formulation of the nonlinear system (2.6) is obtained by introducing a scalar map $v : \mathbb{E} \to \mathbb{R}$ obeying the scalar version of the linear system (2.5). The latter may be used to parametrize the functions ρ^{ik} according to

$$\rho^{ik} = \frac{\Delta_k v^i}{v_i^k - v_k^i}. \tag{2.7}$$

Insertion into (2.6) then produces six equations of which three are independent. These may be written as, for instance,

$$\mathsf{M}\big(v_{ik}^l, v_k^i, v_k^l, v^k, v^l, v^i, v_i^l, v_i^k\big) = 1, \tag{2.8}$$

where the *multi-ratio* of $2n$ numbers is defined by

$$\mathsf{M}_{2n} = \mathsf{M}(a^1, \ldots, a^{2n}) = \frac{(a^1 - a^2)(a^3 - a^4)\cdots(a^{2n-1} - a^{2n})}{(a^2 - a^3)(a^4 - a^5)\cdots(a^{2n} - a^1)}. \tag{2.9}$$

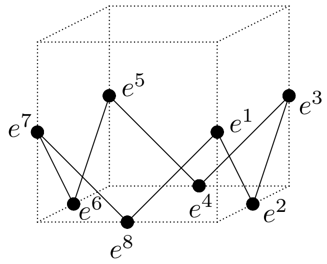

Figure 4. An octagon associated with the M_8 multi-ratio condition.

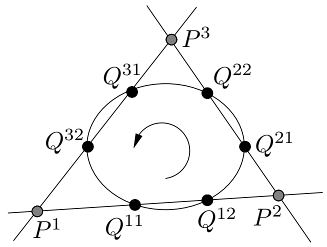

Figure 5. An oriented Carnot figure.

It has been shown in [22] that the origin of these multi-ratio equations resides in a generalization of Menelaus' classical theorem [18, 25]. Each of the six multi-ratio conditions may be associated with one of the six faces of an elementary cube. Indeed, consider the four edges (e^2, e^4, e^6, e^8) of a face as indicated in figure 4 and the four edges (e^1, e^3, e^5, e^7) which are linked to that face. Then, one may regard the edge centres as the vertices of an octagon $(e^1, \ldots, e^8)$ and define an associated multi-ratio condition by

$$\mathsf{M}(v(e^1), \ldots, v(e^8)) = 1. \tag{2.10}$$

Comparison with figure 1 shows that the multi-ratio conditions (2.8) are precisely of the form (2.10).

The additional conic condition defining discrete CKP (dCKP) maps may be expressed in algebraic terms by means of a classical theorem due to Carnot [5], the content of which is the following. Consider an oriented triangle (P^1, P^2, P^3) with two points Q^{n1}, Q^{n2} on each of the (extended) edges (P^n, P^{n+1}) as shown in figure 5 and define an associated product C_6 of ratios of directed lengths according to

$$\mathsf{C}_6 = \prod_{m=1}^{2}\prod_{n=1}^{3} \frac{\overline{P^n Q^{nm}}}{\overline{Q^{nm} P^{n+1}}} \tag{2.11}$$

with the natural identification $P^4 = P^1$. Then, the six points Q^{nm} lie on a conic if and only if

$$\mathsf{C}_6 = 1. \tag{2.12}$$

Even though the orientation of the triangle is irrelevant in the context of Carnot's theorem, it will prove useful in connection with the determination of both a Cauchy problem and a Bäcklund transformation for dCKP lattices $\boldsymbol{v}(\mathbb{E})$.

Carnot's theorem may now be exploited by focusing on one of the four equivalent conic conditions defined on each elementary cube. Thus, we may consider the points $\boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3$ as the vertices of an oriented triangle with the points $\boldsymbol{v}^2_1, \boldsymbol{v}^1_2, \boldsymbol{v}^3_2, \boldsymbol{v}^2_3, \boldsymbol{v}^1_3, \boldsymbol{v}^3_1$ lying on the respective

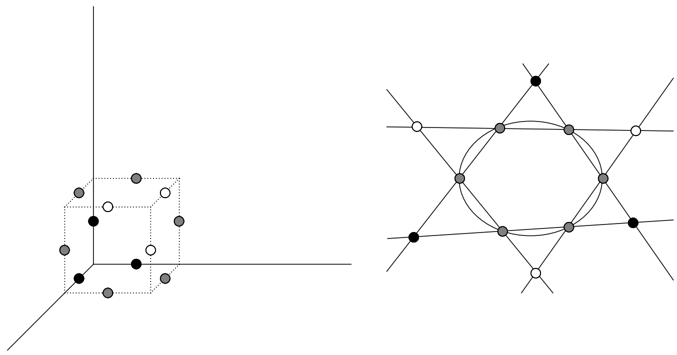

Figure 6. The construction of Cauchy data: $\boldsymbol{v}$ is prescribed on the coordinate axes.

(extended) edges (cf figures 1 and 2). In terms of the coefficients ρ^{ik}, the condition $\mathsf{C}_6 = 1$ then becomes

$$\frac{\rho^{12}(1+\rho^{12})\rho^{23}(1+\rho^{23})\rho^{31}(1+\rho^{31})}{\rho^{21}(1+\rho^{21})\rho^{32}(1+\rho^{32})\rho^{13}(1+\rho^{13})} = 1 \tag{2.13}$$

which, on use of (2.7), translates into a constraint on the scalar map v, namely,

$$\mathsf{M}\big(v^1, v^2_1, v^2, v^3_2, v^3, v^1_3\big) = \mathsf{M}\big(v^1_2, v^2, v^2_3, v^3, v^3_1, v^1\big). \tag{2.14}$$

The latter together with the three lattice equations (2.8) determine all dCKP maps by virtue of the relations (2.7) and the linear system (2.5). *A priori*, it is not evident that the constraint (2.14) is compatible with the well-determined system (2.8). However, in [22], it has been shown that the system (2.8), (2.14) is equivalent to the integrable dCKP equation. The latter encapsulates a discrete version of the classical 'symmetric' Darboux system [9, 21, 23]. In the following, it is shown how the compatibility of the system (2.8), (2.14) may be revealed in a purely geometric manner by determining a well-posed Cauchy problem. Moreover, importantly, it is demonstrated how a Bäcklund transformation for dCKP lattices may be constructed by employing an analogous procedure.

3. A Cauchy problem for dCKP lattices

This section is concerned with the determination of a well-posed Cauchy problem for dCKP maps $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$. To this end, it is convenient to visualize the construction of dCKP lattices by filling the cubic lattice $\mathbb{Z}^3$ with edges. Thus, whenever the value of $\boldsymbol{v}$ on some edge e is known then this edge is inserted in the cubic lattice. We begin by arbitrarily prescribing the map $\boldsymbol{v}$ on the coordinate axes ($n_i = 0, n_k = 0$) as shown in figure 6 and consider the elementary cube which is attached to the origin $(0, 0, 0)$. Here, without loss of generality, we confine ourselves to the construction of dCKP maps which are defined on the first octant of $\mathbb{Z}^3$. The images of the three coordinate edges as indicated by black vertices in figure 6 are the points of intersection of three lines. We now choose a conic which twice intersects each line. The six points of intersection (grey vertices) give rise to another three lines which intersect pairwise (white vertices). Accordingly, by construction, the collinearity and conic conditions are satisfied on the elementary cube.

We proceed by considering an elementary cube which is adjacent to the elementary cube discussed above. The map $\boldsymbol{v}$ is known on a coordinate edge and the four edges of the interface between the two cubes. In figure 7, this fact is indicated by five black vertices, three of which

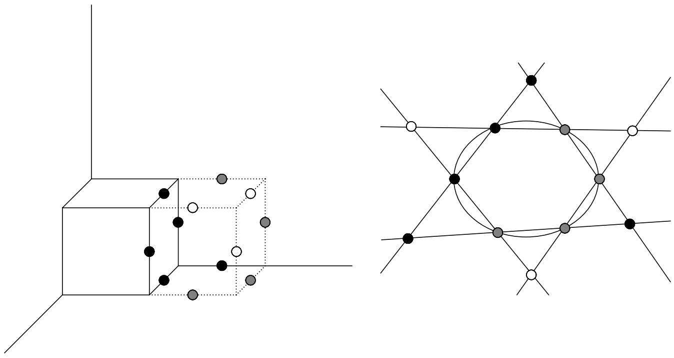

Figure 7. The construction of Cauchy data: $\boldsymbol{v}$ is known on the 'central' elementary cube.

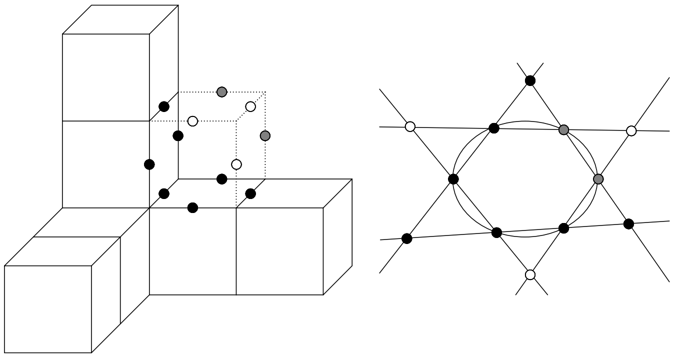

Figure 8. The construction of Cauchy data: $\boldsymbol{v}$ is known on three intersecting cylinders.

form a triangle. If we choose a conic which passes through the other two black vertices on the line corresponding to the interface then we obtain four points of intersection with the remaining two lines (grey vertices). Once again, the picture is completed by drawing three additional lines and determining their points of intersection (white vertices). Iteration of this procedure leads to maps $\boldsymbol{v}$ for which all collinearity and conic conditions are satisfied on three intersecting 'cylinders', that is, on the unmarked cubes in figure 8.

The next step in the procedure is to satisfy the collinearity and conic conditions on an elementary cube which has two faces in common with the above-mentioned cylinders. This situation is illustrated in figure 8. Therein, the two faces correspond to seven black vertices lying on two lines. The two grey vertices on the third line are the points of intersection with a conic which passes through the appropriate four black vertices. As usual, the white vertices are uniquely determined by the remaining three lines. Iteration of this procedure leads to maps $\boldsymbol{v}$ which obey the collinearity and conic conditions on all elementary cubes attached to the three coordinate planes $n_i = 0$ as depicted in figure 9. We denote the corresponding set of edges by

$$\mathbb{E}^0 = \{\boldsymbol{e}^i(n_i = 0), \boldsymbol{e}^i(n_k = 0), \boldsymbol{e}^i(n_k = 1), i \neq k\}. \tag{3.1}$$

A well-posed Cauchy problem is now formulated as follows.

Theorem 1. *A discrete CKP map* $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$ *is uniquely determined by the Cauchy data* $\boldsymbol{v}(\mathbb{E}^0)$ *subject to the collinearity and conic conditions on* $\mathbb{E}^0$.

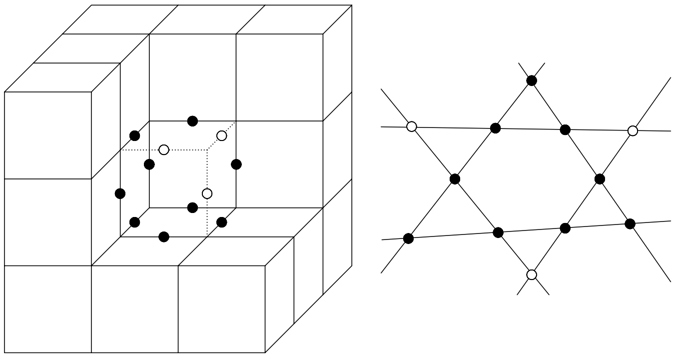

Figure 9. Cauchy data $\boldsymbol{v}(\mathbb{E}^0)$ for discrete CKP maps.

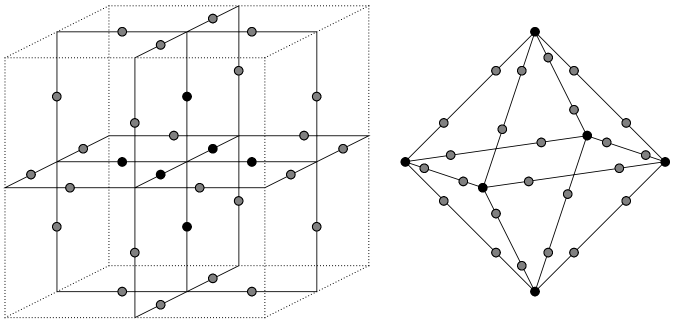

Figure 10. Eight elementary cubes and the associated octahedron.

Proof. In the preceding, it has been shown how one may construct Cauchy data $\boldsymbol{v}(\mathbb{E}^0)$ which obey the collinearity and conic conditions on $\mathbb{E}^0$. We now consider the elementary cube which shares three faces with $\mathbb{E}^0$ as depicted in figure 9. Thus, the map $\boldsymbol{v}$ is known on the nine edges which belong to $\mathbb{E}^0$. Their images are indicated by black vertices. The value of $\boldsymbol{v}$ on the three remaining edges is determined by the points of intersection (white vertices) of the three lines associated with the other three faces of the elementary cube. Accordingly, the collinearity condition holds on the elementary cube. It is evident that iterative application of this procedure leads to a unique discrete Darboux map. Thus, if we consider a cube which consists of eight adjacent elementary cubes as depicted in figure 10 then what we need to demonstrate is that if the conic condition is satisfied on seven elementary cubes then it automatically holds on the eighth cube.

To this end, we focus on the 12 faces which are shared by pairs of elementary cubes. These faces meet at six edges which are indicated by black vertices in figure 10. The four faces linked to any of these edges are mapped to four lines which meet at the image of the common edge. Accordingly, the 6 edges and 12 faces are mapped to the vertices and edges, respectively, of an octahedron as displayed in figure 10. Therein, the grey vertices on the edges of the octahedron represent the images of the remaining 24 edges of the 12 faces. Thus, the conic conditions associated with the eight elementary cubes coincide with the conic conditions on the octahedron. As depicted in figure 11, an incidence theorem of projective geometry, to be discussed below, guarantees that the eight conic conditions are dependent as required. Consequently, the unique discrete Darboux map is indeed of dCKP type. □

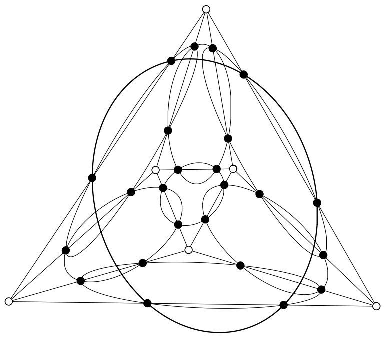

Figure 11. Conic conditions on an octahedron: if seven conic conditions are satisfied then the eighth conic condition automatically holds (bold ellipse).

3.1. A theorem for conics on triangulated surfaces. The origin of integrability

The proof of the above theorem has been based on the fact that the eight conic conditions on an octahedron are not independent. The latter is a particular case of an incidence theorem of projective geometry (cf [19]) which may be formulated as follows.

Theorem 2. *Consider a closed and oriented triangulated surface with two distinct points on each (extended) edge and let N denote the number of triangles. If the conic condition is satisfied on $N-1$ triangles then the conic condition also holds on the Nth triangle.*

Proof. Consider an oriented triangulated surface with a consistent orientation of the triangles, that is, the two orientations of any edge induced by the two adjacent triangles are opposite. Accordingly, a 'Carnot' product C_6^n of the type (2.11) may be associated with each triangle Δ_n. Since every edge makes a contribution to the numerator of one of these products and to the denominator of another product, one obtains the identity

$$\prod_{n=1}^{N} \mathsf{C}_6^n = 1 \tag{3.2}$$

provided that the triangulated surface is closed. Hence, if the conic condition is satisfied on $N-1$ triangles, that is $\mathsf{C}_6^n = 1$ for $n = 1, \ldots, N-1$, then $\mathsf{C}_6^N = 1$ so that the conic condition holds everywhere. □

It has been demonstrated that the above incidence theorem lies at the heart of the Cauchy problem for dCKP lattices. In the next section, it is shown that this theorem also guarantees the existence of a Bäcklund transformation which, in turn, renders dCKP lattices integrable. The connection between discrete integrable systems and 'incidence theorems' has been studied in great detail in the context of 'discrete differential geometry'. For instance, the integrability of so-called curvature lattices, which constitute quadrilateral lattices the faces of which are inscribed in circles, originates in Miquel's classical theorem for six circles meeting at eight vertices [2].

4. A Bäcklund transformation for dCKP lattices

The existence of a Bäcklund transformation for a system of differential or difference equations may be regarded as a definition of integrability (see, e.g., [20]). In the case of the dCKP equation, it is indeed known [9, 21, 22] that one may generate an infinite number of solutions by means of iterative application of an associated Bäcklund transformation to any given seed solution of the dCKP equation. Here, we demonstrate that theorem 2 allows one to construct in a purely geometric manner dCKP maps $\bar{\boldsymbol{v}} : \mathbb{E} \to \mathbb{R}^3$ from any given dCKP map $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$. Thus, the incidence theorem is shown to encode the integrability of discrete CKP maps.

The main idea in the construction of a Bäcklund transformation for dCKP maps is to show that any given dCKP map $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$ may be extended to a map defined on the edges of a $\mathbb{Z}^4$ lattice which obeys the collinearity and conic conditions on all (two-dimensional) faces and (three-dimensional) cubes, respectively. The extended map restricted to the edges of any of the three-dimensional cubic sub-lattices which are 'parallel' to the three-dimensional lattice on which the original dCKP map is defined then constitutes another dCKP map. This idea of higher dimensional 'consistency' has recently been investigated in detail [3, 17]. It may be used efficiently to classify particular classes of integrable discrete equations [1].

In the current context, it is sufficient to focus on maps $\boldsymbol{v}^{(4)}$ which are defined on the set of edges $\mathbb{E}^{(4)}$ of

$$\mathbb{G} = \mathbb{Z}^3 \times \{0, 1\} \subset \mathbb{Z}^4. \tag{4.1}$$

If we denote by $\mathbb{E}$ and $\bar{\mathbb{E}}$ the sets of edges of the two 'horizontal' cubic lattices $\mathbb{Z}^3 \times \{0\}$ and $\mathbb{Z}^3 \times \{1\}$ respectively then, using the decomposition

$$\mathbb{E}^{(4)} = \mathbb{E} \cup \mathbb{E}' \cup \bar{\mathbb{E}}, \tag{4.2}$$

any map

$$\boldsymbol{v}^{(4)} : \mathbb{E}^{(4)} \to \mathbb{R}^3 \tag{4.3}$$

may be split into the maps

$$\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3, \qquad \boldsymbol{v}' : \mathbb{E}' \to \mathbb{R}^3, \qquad \bar{\boldsymbol{v}} : \bar{\mathbb{E}} \to \mathbb{R}^3, \tag{4.4}$$

where $\mathbb{E}'$ designates the set of 'vertical' edges which connect the cubic lattices $\mathbb{Z}^3 \times \{0\}$ and $\mathbb{Z}^3 \times \{1\}$. Thus, if the map $\boldsymbol{v}$ constitutes an arbitrary dCKP map and if it may be extended to a map $\boldsymbol{v}^{(4)}$ in such a way that all collinearity and conic conditions hold on $\mathbb{E}^{(4)}$ then $\bar{\boldsymbol{v}}$ represents another dCKP map.

Once again, we may identify the maps $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\bar{\boldsymbol{v}}$ with maps

$$\boldsymbol{v}^i : \mathbb{Z}^3 \to \mathbb{R}^3, \qquad \boldsymbol{v}^4 : \mathbb{Z}^3 \to \mathbb{R}^3, \qquad \bar{\boldsymbol{v}}^i : \mathbb{Z}^3 \to \mathbb{R}^3 \tag{4.5}$$

respectively, where $i = 1, 2, 3$. For notational convenience, we set $(\boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3) = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, $\boldsymbol{v}^4 = \boldsymbol{d}$ and $(\bar{\boldsymbol{v}}^1, \bar{\boldsymbol{v}}^2, \bar{\boldsymbol{v}}^3) = (\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{c}})$ and label the edges of $\mathbb{G}$ by their images under $\boldsymbol{v}^{(4)}$. Thus, if we denote the 'horizontal' elementary cubes in the cubic lattice $\mathbb{Z}^3 \times \{0\}$ and their 'parallel' counterparts in $\mathbb{Z}^3 \times \{1\}$ by H and $\bar{\mathsf{H}}$, respectively, and the 'vertical' elementary cubes (which link the horizontal cubes) by $\mathsf{V}^1, \mathsf{V}^2, \mathsf{V}^3$ then the edges of the elementary cubes carry the labels

$$\begin{aligned}
\mathsf{H}: &\quad \{\boldsymbol{a}, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_{23}, \boldsymbol{b}, \boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_{13}, \boldsymbol{c}, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_{12}\}\\
\bar{\mathsf{H}}: &\quad \{\bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}_2, \bar{\boldsymbol{a}}_3, \bar{\boldsymbol{a}}_{23}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{b}}_1, \bar{\boldsymbol{b}}_3, \bar{\boldsymbol{b}}_{13}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{c}}_1, \bar{\boldsymbol{c}}_2, \bar{\boldsymbol{c}}_{12}\}\\
\mathsf{V}^1: &\quad \{\boldsymbol{b}, \boldsymbol{b}_3, \bar{\boldsymbol{b}}, \bar{\boldsymbol{b}}_3, \boldsymbol{c}, \boldsymbol{c}_2, \bar{\boldsymbol{c}}, \bar{\boldsymbol{c}}_2, \boldsymbol{d}, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_{23}\}\\
\mathsf{V}^2: &\quad \{\boldsymbol{a}, \boldsymbol{a}_3, \bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}_3, \boldsymbol{c}, \boldsymbol{c}_1, \bar{\boldsymbol{c}}, \bar{\boldsymbol{c}}_1, \boldsymbol{d}, \boldsymbol{d}_1, \boldsymbol{d}_3, \boldsymbol{d}_{13}\}\\
\mathsf{V}^3: &\quad \{\boldsymbol{a}, \boldsymbol{a}_2, \bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}_2, \boldsymbol{b}, \boldsymbol{b}_1, \bar{\boldsymbol{b}}, \bar{\boldsymbol{b}}_1, \boldsymbol{d}, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_{12}\}.
\end{aligned} \tag{4.6}$$

As usual, subscripts i denote elementary shifts along the coordinate direction n_i. Accordingly, any hypercube $\mathsf{H}^{(4)} \subset \mathbb{G}$ is composed of two horizontal cubes $\mathsf{H}, \bar{\mathsf{H}}$ and six vertical cubes $\mathsf{V}^i, \mathsf{V}^i_i$, $i = 1, 2, 3$, as illustrated in figure 12.

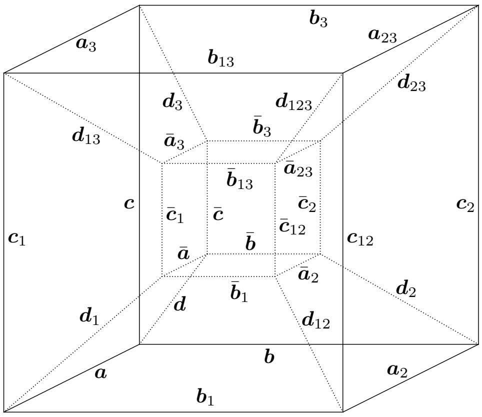

Figure 12. The labelling of a hypercube $\mathsf{H}^{(4)} \subset \mathbb{G}$.

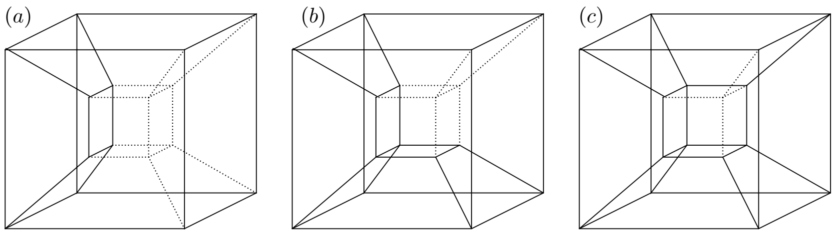

Figure 13. The collinearity and conic conditions are satisfied on the horizontal cube H and on (a) one, (b) two and (c) three vertical cube(s).

4.1. A hypercube theorem

Here, we focus on a single hypercube $\mathsf{H}^{(4)}$ of the lattice $\mathbb{G}$ and demonstrate how the collinearity and conic conditions may be consistently imposed on the eight constituent elementary cubes. In view of the construction of a Bäcklund transformation, we begin with the assumption that the collinearity and conic conditions hold on the horizontal cube H. This is indicated by solid edges in figure 12. If we arbitrarily prescribe a point $\boldsymbol{d}$ which does not lie on the plane associated with the cube H then the four collinear points $\boldsymbol{a}, \boldsymbol{a}_3, \boldsymbol{c}, \boldsymbol{c}_1$ and $\boldsymbol{d}$ span a plane associated with the vertical cube V^2. This situation has been encountered earlier in the construction of the Cauchy problem for dCKP maps and, accordingly, the right-hand side of figure 7 applies. Thus, specification of a suitable conic which passes through the points $\boldsymbol{a}_3$ and $\boldsymbol{c}_1$ gives rise to the points of intersection $\bar{\boldsymbol{a}}, \boldsymbol{d}_1$ and $\bar{\boldsymbol{c}}, \boldsymbol{d}_3$ with the lines $(\boldsymbol{a}, \boldsymbol{d})$ and $(\boldsymbol{c}, \boldsymbol{d})$, respectively. The remaining points $\bar{\boldsymbol{a}}_3, \bar{\boldsymbol{c}}_1$ and $\boldsymbol{d}_{13}$ are then uniquely determined and the collinearity and conic conditions hold on V^2 as indicated by solid edges in figure 13(a).

By construction, the two sets of collinear points $\boldsymbol{a}, \boldsymbol{a}_2, \boldsymbol{b}, \boldsymbol{b}_1$ and $\boldsymbol{a}, \bar{\boldsymbol{a}}, \boldsymbol{d}, \boldsymbol{d}_1$ span a plane which is associated with the vertical cube V^3. This situation is now represented by the right-hand side of figure 8. Thus, a suitably chosen conic which passes through the points $\bar{\boldsymbol{a}}, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{d}_1$ defines the points $\bar{\boldsymbol{b}}$ and $\boldsymbol{d}_2$ which, in turn, may be used to construct the points $\bar{\boldsymbol{a}}_2, \bar{\boldsymbol{b}}_1$ and $\boldsymbol{d}_{12}$. Thus, the collinearity and conic conditions are satisfied on V^3 (cf figure 13(b)).

Finally, the plane associated with the vertical cube V^1 is spanned by the three sets of collinear points $\boldsymbol{b}, \bar{\boldsymbol{b}}, \boldsymbol{d}, \boldsymbol{d}_2, \boldsymbol{b}, \boldsymbol{b}_3, \boldsymbol{c}, \boldsymbol{c}_2$ and $\boldsymbol{c}, \bar{\boldsymbol{c}}, \boldsymbol{d}, \boldsymbol{d}_3$ which uniquely determine the remaining points $\bar{\boldsymbol{b}}_3, \bar{\boldsymbol{c}}_2$, and $\boldsymbol{d}_{23}$ as illustrated by the right-hand side of figure 9. Thus, the collinearity

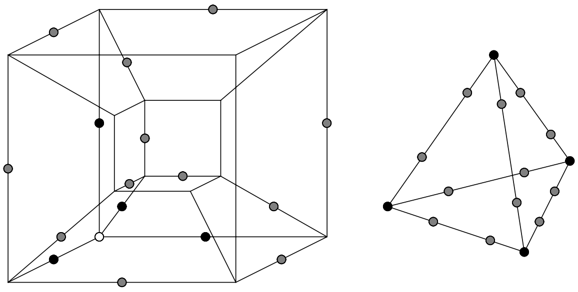

Figure 14. Four cubes and the associated tetrahedron.

conditions hold on V^1 (cf figure 13(*c*)). Remarkably, the associated conic condition is also satisfied due to the following corollary of theorem 2.

Corollary 1. *Consider four adjacent cubes which share a vertex of a hypercube and on which the collinearity conditions are satisfied. If the conic condition holds on three cubes then it automatically holds on the remaining cube.*

Proof. By assumption, the four cubes are linked by four edges (black vertices) which meet at the (white) vertex of the hypercube as illustrated in figure 14. Since each edge is shared by two cubes, the four edges and four cubes are mapped to the vertices and faces, respectively, of a tetrahedron. Moreover, the edges of the hypercube which are marked by grey vertices are mapped to the edges of the tetrahedron. Accordingly, the conic conditions associated with the four cubes coincide with the conic conditions on the tetrahedron. Application of theorem 2 then concludes the proof. □

The preceding analysis now forms the basis of the following *hypercube theorem* which is central to the construction of a Bäcklund transformation for dCKP maps.

Theorem 3. *Consider maps $\boldsymbol{v}^{(4)} : \mathsf{H}^{(4)} \to \mathbb{R}^3$ defined on (the edges of) a single hypercube $\mathsf{H}^{(4)}$. If the collinearity and conic conditions are satisfied on three or four cubes which share a vertex of the hypercube then there exists a unique map $\boldsymbol{v}^{(4)}$ for which the collinearity and conic conditions hold everywhere on $\mathsf{H}^{(4)}$.*

Proof. Let $\boldsymbol{v}^{(4)}$ be initially defined on three cubes which share a vertex of the hypercube $\mathsf{H}^{(4)}$. The latter also constitutes a vertex of a fourth adjacent cube. By assumption, the collinearity and conic conditions are satisfied on the first three cubes. As demonstrated earlier, there exists a unique extension of the map $\boldsymbol{v}^{(4)}$ which obeys the collinearity condition on the fourth cube. Moreover, corollary 1 implies that the conic condition also holds on the fourth cube. Accordingly, in the following, we may assume that the collinearity and conic conditions are satisfied on all four cubes which, for convenience, are taken to be H and $\mathsf{V}^1, \mathsf{V}^2, \mathsf{V}^3$. This means that the collinearity condition is satisfied on 18 of the 24 faces of the hypercube.

Even though the images of the horizontal cube $\bar{\mathsf{H}}$ and the vertical cubes $\mathsf{V}^1_1, \mathsf{V}^2_2, \mathsf{V}^3_3$ have yet to be completely specified, their associated planes, which we denote by $\bar{\mathsf{P}}$ and $\mathsf{P}^1, \mathsf{P}^2, \mathsf{P}^3$, respectively, are determined by the sets of collinear points

$$\begin{aligned}
\bar{\mathsf{P}} &: \quad \{\bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}_2, \bar{\boldsymbol{a}}_3, \bar{\boldsymbol{b}}, \bar{\boldsymbol{b}}_1, \bar{\boldsymbol{b}}_3, \bar{\boldsymbol{c}}, \bar{\boldsymbol{c}}_1, \bar{\boldsymbol{c}}_2\} \\
\mathsf{P}^1 &: \quad \{\boldsymbol{b}_1, \boldsymbol{b}_{13}, \bar{\boldsymbol{b}}_1, \boldsymbol{c}_1, \boldsymbol{c}_{12}, \bar{\boldsymbol{c}}_1, \boldsymbol{d}_1, \boldsymbol{d}_{12}, \boldsymbol{d}_{13}\} \\
\mathsf{P}^2 &: \quad \{\boldsymbol{a}_2, \boldsymbol{a}_{23}, \bar{\boldsymbol{a}}_2, \boldsymbol{c}, \boldsymbol{c}_{12}, \bar{\boldsymbol{c}}_2, \boldsymbol{d}_2, \boldsymbol{d}_{12}, \boldsymbol{d}_{23}\} \\
\mathsf{P}^3 &: \quad \{\boldsymbol{a}_3, \boldsymbol{a}_{23}, \bar{\boldsymbol{a}}_3, \boldsymbol{b}_3, \boldsymbol{b}_{13}, \bar{\boldsymbol{b}}_3, \boldsymbol{d}_3, \boldsymbol{d}_{13}, \boldsymbol{d}_{23}\}.
\end{aligned} \tag{4.7}$$

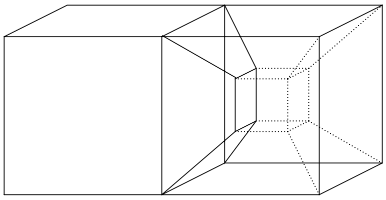

Figure 15. Generation of Cauchy data for the Bäcklund transformation.

Thus, under the assumption of genericity, we may define the points $\bar{\boldsymbol{a}}_{23}$, $\bar{\boldsymbol{b}}_{13}$, $\bar{\boldsymbol{c}}_{12}$ and $\boldsymbol{d}_{123}$ as the points of intersection of triplets of appropriate planes, namely,

$$\begin{aligned} \bar{\boldsymbol{a}}_{23} &= \mathsf{P}^2 \cap \mathsf{P}^3 \cap \bar{\mathsf{P}} \\ \bar{\boldsymbol{b}}_{13} &= \mathsf{P}^1 \cap \mathsf{P}^3 \cap \bar{\mathsf{P}} \\ \bar{\boldsymbol{c}}_{12} &= \mathsf{P}^1 \cap \mathsf{P}^2 \cap \bar{\mathsf{P}} \\ \boldsymbol{d}_{123} &= \mathsf{P}^1 \cap \mathsf{P}^2 \cap \mathsf{P}^3. \end{aligned} \tag{4.8}$$

It is readily seen that the collinearity condition is satisfied on the remaining six faces of the hypercube. Indeed, for instance, the points $\boldsymbol{b}_{13}$, $\bar{\boldsymbol{b}}_{13}$, $\boldsymbol{d}_{13}$, $\boldsymbol{d}_{123}$ are collinear since they lie on both planes P^1 and P^3 by virtue of (4.7) and (4.8). Iterative application of corollary 1 then reveals that the conic condition is satisfied on all eight cubes of $\mathsf{H}^{(4)}$. This completes the proof. □

It is remarked that the proof of theorem 3 shows that the data needed to determine $\boldsymbol{v}^{(4)}$ on a Bäcklund hypercube $\mathsf{H}^{(4)}$ are precisely the vertices on the tetrahedron in figure 14. This observation holds with or without the conic conditions, but it is theorem 2 that guarantees that these data can be chosen to satisfy the conic conditions.

4.2. A Bäcklund transformation

We are now in a position to construct a Bäcklund transformation for dCKP maps. Thus, let $\boldsymbol{v} : \mathbb{E} \to \mathbb{R}^3$ be any given dCKP map with the set $\mathbb{E}$ being the 'horizontal' sub-lattice in the decomposition (4.2) of $\mathbb{E}^{(4)}$. An extension of the map $\boldsymbol{v}$ to a map $\boldsymbol{v}^{(4)} : \mathbb{E}^{(4)} \to \mathbb{R}^3$ which obeys all collinearity and conic conditions is obtained as follows. As demonstrated in the preceding subsection, prescription of the map $\boldsymbol{v}^{(4)}$ on, for example, the vertical cubes V^2 and V^3 of a hypercube $\mathsf{H}^{(4)}$ leads to a unique map of dCKP type defined on all remaining cubes of $\mathsf{H}^{(4)}$. Moreover, since, for instance, the vertical cube V^2_2 is part of both $\mathsf{H}^{(4)}$ and the adjacent hypercube $\mathsf{H}^{(4)}_2$, the collinearity and conic conditions are satisfied on the horizontal cube H_2 and the vertical cube V^2_2 of the hypercube $\mathsf{H}^{(4)}_2$ as illustrated in figure 15. Accordingly, we are in the situation of figure 13(*a*) and may extend the map $\boldsymbol{v}^{(4)}$ in such a way that the collinearity and conic conditions are satisfied on the vertical cube V^3_2. Once again, the hypercube theorem guarantees that the collinearity and conic conditions may be uniquely satisfied on all remaining cubes of $\mathsf{H}^{(4)}_2$. Hence, iterative application of the above procedure leads to a (non-unique) extension of the map $\boldsymbol{v}$ to a map $\boldsymbol{v}^{(4)}$ of dCKP type which is defined on $\mathbb{E}$ and the edges of the set of three intersecting 'cylinders' of hypercubes

$$\hat{\mathbb{E}} = \{\text{edges of } \mathsf{H}^{(4)}(n_i = 0, n_k = 0), i \neq k\}. \tag{4.9}$$

Given any particular extension $\boldsymbol{v}^{(4)}$, we may now construct uniquely another dCKP map $\bar{\boldsymbol{v}}$ which we term a *Bäcklund transform* of the seed dCKP map $\boldsymbol{v}$. The set $\boldsymbol{v}^{(4)}(\hat{\mathbb{E}})$ may be regarded as Cauchy data for the *Bäcklund transformation* which maps $\boldsymbol{v}$ to $\bar{\boldsymbol{v}}$. Here, it is noted

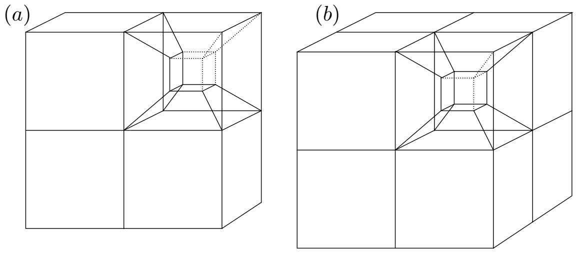

Figure 16. Construction of the unique map $\boldsymbol{v}^{(4)}$.

that the Cauchy data have been generated by imposing the collinearity and conic conditions on a subset of *vertical* cubes only. However, by virtue of the hypercube theorem, these conditions hold everywhere on $\hat{\mathbb{E}}$.

Theorem 4. *Let* $\boldsymbol{v}: \mathbb{E} \to \mathbb{R}^3$ *be a discrete CKP map and let* $\tilde{\boldsymbol{v}}: \mathbb{E} \cup \hat{\mathbb{E}} \to \mathbb{R}^3$ *be an extension of* $\boldsymbol{v}$ *which obeys the collinearity and conic conditions on* $\hat{\mathbb{E}}$*. Then, there exists a unique extension* $\boldsymbol{v}^{(4)}: \mathbb{E}^{(4)} \to \mathbb{R}^3$ *of* $\tilde{\boldsymbol{v}}$ *such that all collinearity and conic conditions are satisfied. In particular, the restriction* $\bar{\boldsymbol{v}}: \bar{\mathbb{E}} \to \mathbb{R}^3$ *constitutes another discrete CKP map.*

Proof. By assumption, the map $\tilde{\boldsymbol{v}}$ obeys all collinearity and conic conditions on the hypercubes $\mathsf{H}^{(4)}(n_i = 0, n_k = 0)$ for $i \neq k$. This set of hypercubes includes the hypercube $\mathsf{H}^{(4)}$ attached to the origin of the coordinate system and its two neighbours $\mathsf{H}_2^{(4)}$ and $\mathsf{H}_3^{(4)}$. Accordingly, the collinearity and conic conditions are satisfied on the vertical cubes V_{23}^2 and V_{23}^3 which are part of the hypercube $\mathsf{H}_{23}^{(4)}$ (cf figure 16(a)). Since, by assumption, these conditions also hold on the horizontal cube H_{23}, the map $\tilde{\boldsymbol{v}}$ may be extended uniquely to the hypercube $\mathsf{H}_{23}^{(4)}$ by virtue of the hypercube theorem. Iterative application of this procedure leads to a dCKP-type extension $\boldsymbol{v}^{(4)}$ of the map $\tilde{\boldsymbol{v}}$ which is defined on $\mathbb{E}$ and the edges of the hypercubes $\mathsf{H}^{(4)}(n_i = 0)$.

We now consider the hypercube $\mathsf{H}^{(4)}$ and its six neighbours $\mathsf{H}_1^{(4)}, \mathsf{H}_2^{(4)}, \mathsf{H}_3^{(4)}$ and $\mathsf{H}_{12}^{(4)}, \mathsf{H}_{23}^{(4)}, \mathsf{H}_{13}^{(4)}$ as illustrated in figure 16(b). The preceding analysis implies that the collinearity and conic conditions hold on the vertical cubes $\mathsf{V}_{123}^1, \mathsf{V}_{123}^2, \mathsf{V}_{123}^3$ and the horizontal cube H_{123} of the hypercube $\mathsf{H}_{123}^{(4)}$. Once again, the hypercube theorem guarantees that the collinearity and conic conditions may be satisfied uniquely on the cubes of H_{123}. Iteration then produces a unique extension $\boldsymbol{v}^{(4)}$ of the map $\tilde{\boldsymbol{v}}$ which is such that the collinearity and conic conditions hold on $\mathbb{E}^{(4)}$. In particular, the map $\boldsymbol{v}^{(4)}$ restricted to the three-dimensional horizontal sub-lattice $\bar{\mathbb{E}}$ constitutes a dCKP map. □

5. Conclusions

We have demonstrated that Pascal's classical theorem for hexagons inscribed in conics allows one to define in a compact manner particular discrete Darboux maps which are governed algebraically by one of the three discrete 'master equations' in soliton theory, namely the dCKP equation. A theorem for conics on closed and oriented triangulated surfaces has been applied to octahedra to construct a well-posed Cauchy problem for dCKP maps. Moreover, the same theorem applied to tetrahedra has led to the construction of a Bäcklund transformation for dCKP maps. Thus, the integrability of dCKP maps and its underlying nonlinear soliton equation has been shown to be encoded entirely in yet another fundamental incidence theorem of projective geometry.

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